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THE TWO-POINT BOUNDARY VALUE PROBLEM IN GEOMETRICAL MECHANICS WITH DISCONTINUOUS FORCES*

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The behavior of mechanical systems in arbitrary finite-dimensional configuration spaces is analyzed using the terminology common to mechanics of geometric-differential formalism (description of virtually all classes of classical mechanics of systems and solid bodies can be found, e.g., in /1—3/).

This paper deals with mechanical systems with discontinuous forces which makes it possible to include in the investigation systems with dry friction, controlled systems, motions in various media, etc. The method of supplementing the definition of discontinuous forces in relation to a multiple-valued vector field with convex images (see, e.g., /4,5/) is used here. It is often applied in classical "plane" situations. The proposed here method of definition supplementing takes into account the specific properties of the nonlinear configuration space, viz. that the sought convex sets in tangent spaces are assumed invariant, i.e. that they are independent of the selection of local coordinate system.

Existence of the system trajectory passing through two specified points of the configuration space (the two-point boundary value problem) is investigated in the case of systems with bounded discontinuous force field.

Note that, unlike in the plane case, this problem is not always solvable even for continuous bounded forces in an arbitrary configuration space. An example of a mechanical system in a two-dimensional sphere with a bounded autonomous smooth force field (independent of velocity) of which not a single trajectory issuing from the south pole reaches the north pole was given in a paper by Gliklikh (**).

The basic assertion of this investigation is that on the assumptions made here there exists a solution of the two-point boundary value problem in some time interval for points of the configuration space that are not conjugate on some geodesic of the Riemann metric that defines the system kinetic energy. A multi-valued operator of the integral type in whose construction the Riemann parallel transfer is used, is derived for the investigation of the problem.

1. Let us consider a discontinuous locally bounded vector field f in a smooth finite-dimensional manifold N, and determine for any point $n_0 \in N$ of the tangential space $T_{n_0}N$ the set $R(n_0)$ as the set of all limit points of the sequence $\{f(n_k)\}$ for any sequence of $n_k \to n_0$, $n_k \neq n_0$. Note that the set $R(n_0)$ is defined by

$$\bigcap_{\varepsilon>0} \{\mathrm{cl}\ [(\bigcup_{n\in U_\varepsilon} f(n))\setminus f(n_0)]\}$$

where U_ε is the ε-neighborhood of point n_0. Let us consider the set $F(n_0) \subset T_{n_0}N$, $F(n_0) = \overline{\mathrm{co}}\, R(n_0)$ ($\overline{\mathrm{co}}$ is the convex closure). We, thus, have obtained the multi-valued mapping of F from N into the tangent stratification TN which associates a convex set in T_nN with every point $n \in N$; it seems appropriate to call that image the multi-valued vector field with convex images.

Note that, when the field f at point n is continuous, $F(n) = f(n)$.

Let us study the properties of mapping $F: N \to TN$. Basic definitions of the theory of multi-valued mapping appear in /5,6/.

Lemma 1. Mapping F is upper semi-continuous.

Proof. Let $n \in N$, $\delta > 0$ be a real number, and ρ some metric on TN that specifies the topology equivalent to the natural topology of the tangent stratification. We denote by $R^\delta(n)$ and $F^\delta(n)$ the δ-neighborhoods in TN of sets $R(n)$ and $F(n)$, respectively, and shall show that for any δ there exists a neighborhood $U(n) \subset N$ of point n such that for every $n' \in U(n)$ we have $R(n') \subset R^\delta(n)$ and, consequently, $F(n') \subset F^\delta(n)$. By the definition of set $R(n)$ there exists an open neighborhood $U(n)$ of point n such that for all $n' \in U(n)$ there exists the distance $\rho(f(n'), R(n)) < \delta$. An open neighborhood $U(n')$ of point n' exists in $U(n)$ such that $\rho(f(n''), R(n')) < \delta$

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* Prikl.Matem.Mekhan.,44,No.3,565-569,1980

**) Iu. E. Gliklikh, Two-point boundary value problem in the geometrical mechanics of systems with bounded force field. Voronezh, 1977. MS No. 2217-77 deposited with VINITI, June 6, 1977.

is satisfied for any $n'' \in U(n')$. We select from $U(n')$ any sequence $n_k'' \to n'$ and obtain

$$\lim \rho(f(n_k''), R(n)) = \rho(\lim f(n_k''), R(n)) < \delta,\ n_k'' \to n'$$

Thus $R(n') \subset R^{\delta}(n)$ and $F(n') \subset F^{\delta}(n)$. The lemma is proved.

2. Let us now consider a mechanical system with quadratic kinetic energy (M, T, α) (see /7/), where the configuration space M is a smooth finite-dimensional manifold; the kinetic energy T is a quadratic function on the tangent stratification (the phase space) TM generated by some Riemannian metric $\langle,\rangle$ in $M: T(X) = {}^1/_2 \langle X, X\rangle$; and the force field $\alpha(t, m, X)$ is a horizontal 1-form on TM which generally depends on time t and on the phase space point (m, X) with $m \in M, X \in T_m M$ (see, also, /8/).

The trajectory of a mechanical system on the phase space TM is an integral curve of a special vector field (second order equation) $\xi = \xi_h(m, X) + \xi_v(t, m, X)$, where $\xi_h(m, X)$ is the geodesic pulverization of coherence of the Levi-Civita metric $\langle,\rangle$, and $\xi_v(t, m, X)$ is the vertical lift of vector $X_\alpha(t, m, X) \in T_m M$ to point X, determined by the equality $\langle X_\alpha, Y\rangle = \alpha(d\pi^{-1}{}_{|X} Y)$for any vector $Y \in T_m M$ (π is the natural projection of TM on M). Vector X_α is correctly determined, since α is a horizontal 1-form.

Expressing the trajectory $\gamma(t)$ on the configuration space in terms of the covariant derivative of coherence of Levi-Civita metric $\langle,\rangle$, we obtain the equation

$$\frac{D}{dt}\gamma^{\cdot}(t) = X_\alpha(t, \gamma(t), \gamma^{\cdot}(t)) \tag{2.1}$$

where $(D/dt$ is the indicated covariant derivative. Equality (2.1) is an invariant formulation of the second Newton's law. Note that any second order differential equation on Riemannian manifold can be represented in the form of an equality of type (2.1) /9/.

We shall consider such mechanical systems whose trajectory does not approach infinity in a finite time interval, when moving by inertia, i.e. in the absence of a force field. In terms of Riemannian goemetry this means that the metric $\langle,\rangle$ is total. We impose on field α the requirement of boundedness $\|\alpha\|$ in some neighborhood of every point (t, m, X) without assuming the continuity with respect to any of the variables.

In investigating the behavior of the mechanical field trajectory we use the construction described in Sect.1 for passing from the discontinuous vector field $\xi_h + \xi_v$ on TM to the multi-valued upper semi-continuous vector field with convex images. For the vector field $\xi_h + \xi_v$ on TM the corresponding multi-valued field is of the form $\xi_h + \Xi_v^{\alpha}(t, m, X)$, where $\Xi_v^{\alpha}(t, m, X)$ is the vertical lift to point X of the set with $Q(t, m, X) = \Xi_\alpha(t, m, X)$, , and $Q(t, m, X)$ is the set of all limit points of sequences $\{X_\alpha(t_k, m_k, X_k)\}$ for any sequence $(t_k, m_k, X_k) \to (t, m, X), (t_k, m_k, X_k) \neq (t, m, X)$.

We thus obtain the second order differential inclusion on manifold M

$$\frac{D}{dt}\gamma^{\cdot}(t) \in \Xi_\alpha(t, \gamma(t), \gamma^{\cdot}(t)) \tag{2.2}$$

with convex images, which describes the trajectory of a system with discontinuous force field in the configuration space.

The C^1 curve $\gamma(t)$ such that function $\gamma^{\cdot}(t)$ is absolutely continuous and for $D\gamma^{\cdot}(t)/dt$ the inclusion (2.2) is almost everywhere satisfied, is called the solution of inclusion (2.2).

The local theorem on the existence of solution of inclusion (2.2) follows the similar theorem for first order differential inclusions in a linear space (see, e.g., /5/).

Note that from the mechanical point of view of interest is the case when the solution of the differential inclusion is locally unique. Some uniqueness theorems were obtained in /4/.

3. To investigate the over-all behavior of solutions of inclusion (2.2) on the complete Riemannian manifold M we construct an operator of the integral type for whose determination the Riemann parallel transfer is used.

Let $w: [0, t_0] \to T_{m_0} M$ be a continuous curve in the tangential space $T_{m_0} M$, $m_0 \in M$. We denote by $Sw(t)$ the C^1 curve in the manifold M, such that $Sw(0) = m_0$ and the vector $dSw(t)/dt \in T_{Sw(t)} M$ is for all t parallel to vector $w(t) \in T_{m_0} M$ along the curve Sw itself. Curve Sw exists and is unique.

Indeed

$$Sw(t) = \delta^{-1} \int_0^t w(\tau)\, d\tau$$

where δ is the Cartan involute /10/. Since M is a complete Riemannian manifold, curve Sw is determinate on the whole interval $[0, t_0]$ (see /9/). The operator S has been, thus, correctly determined /9,11/. It transfers the Banach space $C^0([0, t_0], T_{m_0} M)$ of continuous curves in $T_{m_0} M$ into the Banach manifold $C^1([0, t_0], M)$ of C^1-curves in M.

Let us specify the properties of operator S which will be subsequently required. Detailed proofs of these statements were given earlier (see footnote on p. 396).

Lemma 2. Operator $S : C^\circ([0, t_0], T_{m_0}M) \to C^1([0, t_0], M)$ is continuous.

Lemma 3. The inequality $\|dSw(t)/dt\| \leqslant k$ is satisfied for any curve $w(t)$ from the sphere U_k of radius $k > 0$ whose center is at the zero of space $C^\circ([0, t_0], T_{m_0}M)$ for all $t \in [0, t_0]$.

This statement follows from that the Riemann parallel transfer does not change the norm of a vector.

Lemma 4. Let point $m_1 \in M$ be not a conjugate of m_0 along some geodesic of metric $\langle\ ,\ \rangle$. For any geodesic $a(t), a(0) = m_0, a(1) = m_1$ along which m_0 and m_1 are not conjugate and any number $k > 0$ there exists a number $L(m_0, m_1, k, a) > 0$ such that when $0 < t_1 < L(m_0, m_1, m, a)$ a unique vector $C_w \in T_{m_0}M$ continuously dependent on w and such that $S(w + C_{w_0})(t_1) = m_1$ can be found for any curve $w(t) \in U_k \subset C^\circ([0, t_1], T_{m_0}M)$ in some bounded neighborhood of vector $t_1^{-1}a^\cdot(0) \in T_{m_0}M$.

Let us consider the multi-valued vector field $\Xi_\alpha(t, \gamma(t), \gamma^\cdot(t))$ along the C^1-curve $\gamma(t) = Sw(t)$. We transfer all sets Ξ_α parallel with γ to point $m_0 = \gamma(0)$ which for fixed w yields the multi-valued mapping $\Gamma\circ Sw$ from segment $[0, t_0]$ into $T_{m_0}M$ with convex images. By using the properties of parallel transfer and the upper semi-continuity of field $\Xi_\alpha(t, m, X)$ it can be shown that the mapping $\Gamma\circ S : C^\circ([0, t_0], T_{m_0}M) \times [0, t_0] \to T_{m_0}M$ is upper semi-continuous. Let us consider the set of all measurable cross sections $P\Gamma\circ Sw$ of the multi-valued mapping $\Gamma\circ Sw : [0, t_0] \to T_{m_0}M$, that are known to exist /12/. The local boundedness of field X_α and compactness of curve $\gamma = Sw$ in $[0, t_0]$ imply that all curves from $P\Gamma\circ Sw$ are bounded, i.e. integrable. It is now possible to determine the multi-valued mapping $\int P\Gamma\circ S$ in terms of convex images in the Banach space $C^\circ([0, t_0], T_{m_0}M)$

$$\int P\Gamma\circ S(w) = \left\{\int_0^t v(\tau)\,d\tau \mid v \in P\Gamma\circ Sw\right\}$$

Lemma 5. Mapping $\int P\Gamma\circ S$ transforms bounded sets $C^\circ([0, t_0], T_{m_0}M)$ into compact ones.

Proof. Lemma 3 and the completeness of metric $\langle\ ,\ \rangle$ imply that for any sphere $U_k \subset C^\circ([0, t_0], T_{m_0}M)$ the set of curves $\{(\gamma, \gamma^\cdot) \mid \gamma \in SU_k\}$ lies in the compact set of manifold TM. It follows then from the local boundedness of $X_\alpha(t, m, X)$ that all sets $\Xi_\alpha(t, \gamma(t), \gamma^\cdot(t)), \gamma \in SU_k$ are uniformly bounded. Since the parallel transfer does not alter the norm of a vector, all sets $(\Gamma\circ Sw)(t), w \in U_k$ and their measurable cross sections $P\Gamma\circ Sw$ are uniformly bounded. Hence all continuous curves

$$v \in \bigcup_{w \in U_k} \left(\int P\Gamma\circ S\right) w$$

are uniformly bounded and equicontinuous. The lemma is proved.

Lemma 6. Mapping $\int P\Gamma\circ S$ is upper semi-continuous.

Proof. It is sufficient to show that the multi-valued mapping $\int P\Gamma\circ S$ has a closed diagram, i.e. that as $w_k \to w_0$ and when $v_k \in (\int P\Gamma\circ S)\, w_k, v_k \to v_0$, the inclusion $v_0 \in (\int P\Gamma\circ S) w_0$ is satisfied or, what is equivalent, that inclusion $v_0^\cdot(t) \in (\Gamma\circ Sw_0)(t)$ is satisfied for almost all t. Since mapping $\int P\Gamma\circ S$ transforms bounded sets into compact ones, the closeness of the diagram implies upper semi-continuity /6/. Inclusion $v_0^\cdot(t) \in (\Gamma\circ Sw_0)(t)$ follows from the convexity of sets $(\Gamma\circ Sw_0)(t)$ and upper semi-continuity of mapping $(\Gamma\circ Sw)(t)$ with respect to w and t. A detailed proof a similar inclusion in a simpler case is given in /13/.

4. Let us now formulate and prove the fundamental statement.

Theorem. We assume that point m_1 is not a conjugate of point m_0 along some geodesic a of the metric $\langle\ ,\ \rangle$ and that field $\alpha(t, m, X)$ is by norm uniformly bounded for all t, m, X. There exists such a number $L(m_0, m_1, a)$ that for any $t_0, 0 < t_0 < L(m_0, m_1, a)$ a solution $\gamma(t)$ of inclusion (2.2), such that $\gamma(0) = m_0$ and $\gamma(t_0) = m_1$ can be found.

Proof. Since the field $\alpha(t, m, X)$ is uniformly bounded, the multi-valued vector field $\Xi_\alpha(t, m, X)$ is also uniformly bound by some number $k > 0$. It can be shown that for fairly small $t_1 > 0$ the inequality $t_1 < L(m_0, m_1, kt_1, a)$, where $L(m_0, m_1, kt_1, a)$ is a number from Lemma 4, is satisfied.

We determine the number $L(m_0, m_1, a)$ by the equality $L(m_0, m_1, a) = \sup t_1$ such that $t_1 < L(m_0, m_1, kt_1, a)$.

Let us consider the multi-valued upper semi-continuous compact mapping $Aw = \int P\Gamma\circ S(w + C_w)$, where C_w is a vector from Lemma 4, on the sphere $U_{kt_0} \subset C^\circ([0, t_0], T_{m_0}M)$. Since the parallel transfer does not alter the norm of a vector, A transforms U_{kt_0} into itself and, consequently, has always a fixed point $w_0, w_0 \in Aw_0$ /14/.

Let us show that $\gamma_0 = S(w_0 + C_{w_0})$ is the sought solution of (2.2). By construction $\gamma_0(0) = m_0$, $\gamma_0(t_0) = m_1$, γ_0 is the C^1 curve, and function $\gamma_0^\cdot$ is absolutely continuous. Since w_0 is the fixed point of A, hence $w_0^\cdot$ is the cross section $\Gamma\circ S(w_0 + C_{w_0})$, i.e. at points t, where

$w_0^{\cdot}(t)$ exists, the inclusion $w_0^{\cdot}(t) \in \Gamma \circ S(w_0 + C_{w_0})(t)$ is satisfied. By the construction and properties of the covariant derivative, after the parallel transfer of $w_0^{\cdot}(t)$ and $\Gamma \circ S(w_0 + C_{w_0})(t)$ along γ_0 to point $\gamma_0(t)$ we obtain, respectively, $D\gamma_0^{\cdot}(t)/dt$ and $\Xi_\alpha(t, \gamma_0(t), \gamma_0^{\cdot}(t))$. Thus $D\gamma_0^{\cdot}(t)/dt \in \Xi_\alpha(t, \gamma_0(t), \gamma_0^{\cdot}(t))$. The theorem is proved.

Note that when m_0 and m_1 are not conjugate along several geodesics, any of them can be used for proving the existence of solution. The numbers L and the solutions themselves constructed on different geodesics are generally different.

If the configuration space M (with metric $\langle , \rangle$) is a compact manifold of nonpositive curvature and the force field bounded, there exists a number $L > 0$ such that for any points m_0 and m_1 and any time $[0, t_0], 0 < t_0 < L$ the two-point boundary value problem is solvable. This is the result of absence of conjugate points and compactness of the manifold.

When M is a plane manifold, it is possible to show that the number L from Lemma 4 is infinite, i.e. we obtain the known result that in the plane case with bounded force field the two-point boundary value problem is solvable for any two points and any time interval.

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